Program : M.A./M.Sc. (Mathematics)
M.A./M.Sc. (Final)
Paper Code:MT-09
Integral Transforms and Integral Equations
Section – C
(Long Answers Questions)

1. Prove that $L[J_0(t); P] = \frac{1}{\sqrt{1+p^2}}$ and hence deduce that $L\{e^{-at}J_0(bt); p\} = \frac{1}{\sqrt{p^2+2ap+a^2+b^2}}$
2. If $f(t)$ is a periodic function with period $T > o$ then derive the Laplace transform if (t) ; also define periodic function.
3. Obtain $L[er\,f(t); P]$ hence deduce the value of $L[er\,f(bt); p]$
4. Prove that $L\left[\frac{sin^2 t}{t}\ ip\right] = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$ and deduce that :
   (i) $\int_0^\infty e^{-t}\frac{sin^2 t}{t}dt = \frac{1}{4}\log 5$
   (ii) $\int_0^\infty \frac{sin^2 t}{t^2}dt = \frac{\pi}{2}$
5. Evaluate : $L\left[\frac{1-\cos t}{t^2}\ ; p\right]$
6. Prove that $L\left\{\frac{\cos at - \cos bt}{t}\ ; p\right\} = \frac{1}{2}log\left(\frac{p^2+b^2}{p^2+a^2}\right)$ hence deduce that $\int_0^\infty \left[\frac{\cos at - \cos bt}{t}\right]dt = \log\frac{b}{a}$
7. Using partial fractions. Find $L^{-1}\left[\frac{p^2}{p^4+4a^4}\right]$
8. Define convolution of two functions and prove that If $f(t)$ and $g\,(t)$ are two functions of class A of and if $L^{-1}[\bar{f}\,(p);\ t] = f(t); L^{-1}[\bar{g}\,(p);\ t] = g(t)$, then $L^{-1}[\bar{f}\,(p).\bar{g}\,(p);\ t] = \int_0^t f(u)g(t-u)du = f * g$
9. Apply convolution theorem to prove that $B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, (m > 0, n > 0)$ hence deduce that: $\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$ where B9m, n) is called Beta function.
10. Use complex inversion formula to obtain the inverse Laplace transform of $\frac{p}{(p+1)(p-1)^2}$
11. Find $L^{-1}\left[\frac{\cosh u\sqrt{P}}{P cosh\sqrt{P}}\right]$ where 0 < u < 1.
12. Find $L^{-1}\left[\frac{3P-1}{p(p-1)^2 9P+1)}\right]$ be complex inversion formula.
13. Solve : $ty'' + (t-1)y' - y = 0, y(0) = 5, y(\infty) = 0.$
14. A semi infinite rod $x > o$ is initially at temperature zero. At time $t > 0$ a constant temperature $V_0 = 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at any time t > 0.

15. An infinite long string having one end $x = 0$ is initially at rest on the x-axis. The end, $x = 0$ undergoes a periodic transverse displacement given by $\Delta_0 \sin wt, t > 0$. Find the displacement of any point on the string at any time.
16. A flexible string has its end points on the x-axis at $x = 0$ and $x = c$. At time $t = 0$, the string is given a shape defined by $b \sin\left(\frac{\pi x}{c}\right), 0 < x < c$ and released. Find the displacement of any point x of the string at any time $t > 0$.
17. Find the solution of the equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ which tends to zero as $x \to \infty$ and which satisfies the conditions :
$$u = f(t) at\ x = 0, t > 0 \ \&\ u = 0\ at\ x > 0, t = 0$$
18. Find the solution of Diffusion equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0$ subject to the initial and boundary conditions $u\,(x, 0) = 0, x > 0\ ;\ -K\left(\frac{\partial u}{\partial x}\right) = f(t) at\ x = 0, t > 0\ and\ u\,(x,t) \to 0\ as\ x \to \infty\ and\ t > 0$ where k & k are respectively the thermal diffusivity and conductivity of material of given solid.
19. Find the fourier transform of $f(t)$, where $f(t) = \begin{cases} 1 - t^2, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$ and hence evaluate $\int_0^\infty \left(\frac{t\cos t - \sin t}{t^3}\right) \cos\frac{t}{2}\, dt$
20. Find $f(t)$ if its fourier sine transform is $\frac{P}{(1+p)^2}$
21. Prove that $e^{-t2/2}$ is a self-reciprocal function under the fourier cosine transform. Hence obtain the fourier sine transform of $\left(te^{-t2/2}\right)$
22. State and prove convolution theorem for fourier transform.
23. Using Parseval's Identity prove that:
(i) $\int_{-\infty}^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} \frac{\pi}{2ab\,(a+b)}, (a > 0, b > 0)$
(ii) $\int_{-\infty}^{\infty} \frac{\sin at}{(a^2+t^2)} dt = \frac{\pi}{2}\left\{\frac{1-e^{-a^2}}{a^2}\right\}$
24. Evaluate : $\int_{-\infty}^{\infty} \frac{dt}{(t^2+a^2)(t^2-b^2)}, a > 0, b > 0$
25. Prove that $M\,\{e^{-ax} J_2(bx); p\} = \frac{b^v 2^{p-1}}{\sqrt{\pi}\Gamma(v+1)} (a^2 + b^2)^{\frac{-(v+p)}{2}}$
$\Gamma\left(\frac{v+p}{2}\right) \Gamma\left(\frac{v+p+1}{2}\right) 2f_1 \left[\frac{v+p}{2}, \frac{v-p+1}{2}; v+1; \frac{b^2}{a^2+b^2}\right]$
$(Re(a) > 0, u < -\frac{1}{2}$
Hence deduce that
(i) $M\{J_v(bx); p\} = \frac{b^{-p} 2^{p-1} \Gamma\left(\frac{v+p}{2}\right)}{\Gamma\left(\frac{v-p+2}{2}\right)}$; -v < p < v+2
(ii) $M\{x^{-v} J_v(x); p\} = \frac{2^{p-v-1}\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(v - \frac{1}{2}p + 1\right)}$; $o < Re\,(p) < 1, v > -\frac{1}{2}$
26. Prove that :
$$M\{x^\rho (1-x)^{c-1} 2f_1(a, b\,; c\,; 1-x) H\,(1-x); p\} = \frac{\Gamma(c)\Gamma(p+\rho)\Gamma(p-a-b+c+\rho)}{\Gamma(p-a+c+\rho)\Gamma(p-b+c+\rho)}$$
27. Prove that if m is a positive integer, $\alpha \neq 0$

$$M\left\{\left(x^{1-\alpha}\frac{d}{dx}\right)^m f(x);p\right\} = (-1)^m \alpha^m \frac{\Gamma(\frac{p}{\alpha})}{\Gamma(\frac{p}{\alpha}-m)} f(p-ma)$$

$where\ M\{f(x0\,;p\} = f(p)$

28. If $F(p)$ and $G(p)$ are Mellin transform of $f(x)$ and $g(x)$ respectively find the mellin transform of :

$x^\lambda \int_0^\infty u^\mu f\left(\frac{x}{u}\right) g(u)du$ where λ and μ are constants.

29. Obtain the Mellin transform of $f(x) = \frac{(1-x^2)^{\lambda-1} H(1-x)}{\Gamma(\lambda)}$

$g(x) = \frac{2(1-a^2x^2)^{\mu-1} H(1-ax)}{\Gamma(\mu)}\ with\ \lambda < 0, \mu > 0, o < a < 1$ hence or otherwise establish that

$$\frac{1}{2\pi_2}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{z}{2}\right)\Gamma\left(\alpha-\frac{z}{2}\right)a^z}{\Gamma\left(\beta+\frac{z}{2}\right)\Gamma\left(\gamma-\frac{z}{2}\right)} dz = \frac{2a^{2\alpha}}{\Gamma(\alpha+\beta)\Gamma(\gamma-\alpha)} 2F_1\begin{bmatrix}\alpha, \alpha+1-\gamma; a^2\\ \alpha+\beta;\end{bmatrix}$$

$with\ o < \alpha \le 1, 0 < \alpha < \gamma, \beta > 0.$

30. Find the Mellin transform of sin x and show that :

$$M^{-1}\left\{\Gamma(p)\sin\left(\frac{p\pi}{2}\right)f^*(1-p);x\right\} = \sqrt{\frac{\pi}{2}}F_s\{f(t);x\}$$

$where\ f^*(p) = M\{f(t);p\}$

31. Prove that if $v > -\frac{1}{2}$ then

$$H_v\{x^{v-1}e^{-ax};p\} = L\{x^v J_v(px);a\} = \frac{2^v p^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(a^2+p^2)^{v+\frac{1}{2}}}$$

32. Prove that :

$H_v\{e^{-px2/4} f(x);s\} = 2L\{f(2\sqrt{x}J_v(2s\sqrt{x});p\}$

Deduce that $H_v\left\{x^v e^{-p\frac{x^2}{4}};s\right\} = \frac{2^{v+1}s^v}{p^{v+1}} e^{-\frac{s^2}{p}}$

and hence that:

(i) $H_v\left\{x^v e^{-\frac{x^2}{a^2}};s\right\} = \left(\frac{a^2}{2}\right)^{v+1} e^{-a^2\frac{s^2}{4}}$

(ii) $H_v\left\{x^v e^{-\frac{x^2}{2}};s\right\} = s^v e^{-\frac{s^2}{2}}$

33. Prove that :

$H_v\{x^v(a^2-x^2)^{\mu-v-1} \cup (a-x);p\} = 2^{\mu-v-1}$
$\Gamma(\mu-v)P^{v>\mu}a^\mu J_\mu(pa), a > 0, \mu > v > 0$

Hence deduce:

(i) $H_v\{x^v \cup (a-x);p\} = \frac{a^{v+1}}{P} J_{v+1}(pa), a > 0\ and$

(ii) $H_v\left\{\frac{x^v \cup (a-x)}{\sqrt{a^2-x^2}};p\right\} = \sqrt{\frac{\pi}{2p}} a^{\frac{a^{v+1}}{2}} J_{v+\frac{1}{2}}(p,a)$

34. Prove that :

$$H_v\{x^{v-\mu}J_\mu(a,x);p\} = \frac{p^v(a^2-p^2)^{\mu-v-1}}{2^{\mu-v-1}\Gamma(\mu-v)a^H} \cup (a-p)(a > 0, \mu > v \ge 0)$$

Deduce that:

(i) $H_v\{x^{-1}J_{v+1}(a,x);p\} = \frac{p^v}{a^{v+1}} \cup (a-p); a > 0$

(ii) $H_v\left\{x^{-\frac{v}{2}}J_{v+\frac{1}{2}}(ax);p\right\} = \sqrt{\frac{2}{\pi}}\frac{p^v \cup(a-p)}{a^{v+\frac{1}{2}}(a^2-p^2)^{\frac{1}{2}}}, a > 0, v \geq 0$ and hence that

(iii) $H_0\{x^{-2}(1 - J_0(a,x); P)\} = H\,(a-p)log\left(\frac{a}{p}\right)$

35. If $f(x) = \frac{e^{-ax}}{x}$, then find (i) the Hankel transform of order zero of the function $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}$ and (ii) the Hankel transform of order one of $\frac{df}{dx}$.

36. Find the Hankel transform of $x^v H(a-x) and\, x^v\, H(b-x), v > -\frac{1}{2}$. Hence or otherwise establish that:

$$H_v\{x^{-2}J_v(a,x);p\} = \begin{cases} \frac{1}{2v}(\frac{p}{a})^v, 0 < p < a \\ \frac{1}{2v}(\frac{a}{p})^v\ ; p > a \end{cases}$$

37. Solve the Laplace equation in the half plane :

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, (-\infty < a < \infty, y \geq 0)$$

With the boundary conditions:

$U(x,0) = f(x), \quad -\infty < x < \infty$

and $U(a,y) \to 0\ as\ |x| \to \infty, y \to \infty$

38. Show that the solution of Laplace equation for U inside the semi-infinite strip $x > 0, o < y < b$ such that:

$U = f(x), where\ y = 0, 0 < x < \infty$

$U = 0, where\ y = b, 0 < x < \infty$

$U = 0, where\ x = 0, 0 < y < b$

Is given by $U = \frac{2}{\pi}\int_0^\infty f(u)du \int_0^\infty \frac{\sin h\,(b-y)p}{\sin h\, pb} \sin xp \sin up\; dp$

39. Heat is supplied at a constant rate Q per in the plane $z = 0$ to an infinite solid of conductivity K. Show that the steady temperature at a point distant r from the axis of the circular area and distance z from the late $r = 0$ is given by:

$$\frac{Qa}{2k}\int_0^\infty e^{-pz}J_0(pr)J_1(pa)p^{-1}dp$$

40. The free symmetric vibrations of a very large membrane are governed by the equation:

$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}, \quad r > 0, t > 0$ with $U = f(t), \frac{\partial U}{\partial r} = g(r), t = 0$

$show\ that\ for\ t > 0$

$$U(r,t) = \int_0^\infty PF(p)\cos(pct)J_0(pr)dp + \frac{1}{c}\int_0^\infty G(p)\sin(pct)J_0(pr)dp$$

Where $f(p)$ and $G(p)$ are the zero order Hankel transforms of f® and g® respectively.

41. Find the potential V9r, z) of a field due to a flat circular disc of unit radius with its centre at the origin and axis along the z-axis satisfying the differ initial equation:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \qquad o \leq r \leq \infty, z \geq 0$$

and satisfying the boundary conditions :

$V = V_0$ *when* $z = 0, 0 \leq r < 1$ *and* $\frac{\partial v}{\partial z} = 0$, *when* $z = 0, e > 1$

42. Solve the initial value problem for the wave equation $\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$, $(-\infty < x < \infty, t > 0)$ subject to condition
$U(x, o) = f(x)$
$U_t(x, 0) = g(x), (-\infty < x < \infty)$
43. Show that the function $g(x) = xe^x$ is a solution of the volterra integral equation:
$$g(x) = \sin x + 2\int_0^x \cos(x-t) g(t) dt$$
44. Show that the function $g(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of the Fredholm integral equation.
$$g(x) - \frac{\pi^2}{4}\int_0^1 K(x,t) g(t) dt = \frac{x}{z}$$
45. Form a integral equation corresponding to the differential equation :
$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + (x^2 - x)y = xe^x + 1$$
With conditions : $y(0) = 1 = y'(0) and\ y''(0) = 0$
46. Reduce the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4\sin x$ with the conditions $y(0) = 1, y'(0) = -2$ into a non homogeneous Volterra's integral equation of second kind. Conversely derive the original differential equation with the initial conditions from the integral equation obtained.
47. Convert the differential equation $\frac{d^2y}{dx^2} + \lambda y = 0$ with the conditions $y(0) = 0, y(l) = 0$ into fredholm integral equation of second kind. Also recover the original differential equation from the integral equation you obtain.
48. Prove that the characteristic numbers of a symmetric kernel are real.
49. Find the eigen values and eigen function of the homogeneuous integral equation:
$$g(x) = \lambda\int_0^\pi [cos^2x \cos 2t + \cos 3x\ cos^3t]\ g(t) dt$$
50. Solve the $f(x)$ the integral equation:
$$\int_0^\infty f(x0 \cos p\ d\ dx = \begin{bmatrix} 1 - p, 0 \leq p \leq 1 \\ 0, p > 1 \end{bmatrix}$$
Hence deduce that $\int_0^\infty \frac{sin^2 t}{t^2} dt = \frac{\pi}{2}$
51. Find the resolvent kernel of the volterra integral equation and hence its solution.
$$g(x) = f(x) + \int_0^x (x-t) g(t) dt$$
52. Solve the integral equation:
$$g(x) = e^{-x} - 2\int_0^x \cos(x-t) g(t) dt$$
53. Solve the Abel integral equation:

(i) $f(x) = \int_0^x \frac{g(t)}{(x-t)^\alpha}\, dt, 0 < x < 1$

(ii) $\int_0^x \frac{g(t)}{\sqrt{x-t}} dt = 1 + x + x^2$

54. Solve the integral equation and discuss all its possible cases by the method of degenerate kernels :

$g(x) = f(x) + \lambda \int_0^1 (1 - 3xt)\, g(t)dt$

55. Solve the integral equation:

$g(x) = x + \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin t)\; g(t)dt$

56. Solve the fredholm integral equation of second kind.

$$g(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)\, g(t)dt$$

57. Find the resolvent kernels of the following kernels:

(i) $k(x,t) = (1+x)(1-t), a = -1, b = 0$

(ii) $k(x,t) = e^{x+t}, \quad a = 0, b = 1$

58. Solve by the method of successive approximation:

$$g(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2} + \frac{1}{2}\int_0^1 t\, g(t)dt$$

59. By iterative method solve:

$$g(x) = 1 + \lambda \int_0^{\pi} \sin(x+t) g(t)dt$$

60. Find the resolvent kernel of the following integral equation :

$$g(x) = 1 + \lambda \int_0^1 (1 - 3xt) g(t)dt$$

61. Find the resolvent kernel of the Volterra integral equation with the kernel.

$$k(x,t) = \frac{(2 + \cos x)}{(2 + \cos t)}$$

62. Solve $g(x) = \cos x - x - 2 + \int_0^x (t - x) g(t)dt$

63. If a kernel is symmetric then show tat all its iterated kernels are also symmetric.

64. Solve the symmetric integral equation.

$$g(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2) g(t)\, dt$$

65. Solve the following symmetric integral equation with the help of Hilbert-schmidt theorem.

$$g(x) = 1 + \lambda \int_0^{\pi} \cos(x+t) g(t)dt$$

66. Using Hilbert-Schmidt method, solve integral equation

$$g(x) = 1 + \lambda \int_0^1 \mathrm{k(x,t)}\, g(t)dt$$

Where $K(x,t) = \begin{cases} x\,(t-1); 0 \le x \le t \\ t(x-1), t \le x \le 1 \end{cases}$

67. State and prove Hilbert-Schmidt theorem.

68. Using Hilbert-Schmidt theorem, solve integral equation:

$g(x) = \cos \pi x + \lambda \int_0^1 k(x,t) g(t)dt$ where

$$K(x,t) = \begin{cases} (\lambda + 1)t, & 0 \le x \le t \\ (t+1)x, & t \le x \le 1 \end{cases}$$

69. Using fredholm's determinants find the resolvent kernel of the following kernel.

$\sin x \cos t\,,\quad 0 \le x \le 2\pi,\quad 0 \le t \le 2\pi$

70. Solve the integral equations :

$$g(x) = 1 + \lambda \int_0^{\pi} \sin(x+t) g(t)dt$$

71. Find $D(\lambda)$ and $D(x,t\,;\,\lambda)$ ans dolve the integral equation.

$$g(x) = x + \lambda \int_0^{1} \left[xt + \sqrt{xt}\right] g(t)dt$$

72. Using fredholm determinants find the resolvent kernels, when $k(x,t) = x\,e^t, a = 0, b = 1.$

73. Find the resolvent kernel and solution of

$$g(x) = f(x) + \lambda \int_0^{1} (x+t) g(t)dt$$

74. Using securrence relations find the resolvent kernels of the following kernels:

(i) $k(x,t) = \sin x \cos t\,; 0 \le x \le 2\pi,\ 0 \le t \le 2\pi$

(ii) $k(x,t) = 4xt - x^2;\ 0 \le x \le\le 1,\ 0 \le t \le 1$