Program : M.A./M.Sc. (Mathematics)

M.A./M.Sc. (Final)

Paper Code:MT-06

Analysis and Advanced Calculus

Section – C

(Long Answers Questions)

1. Let p > 1 and $\frac{1}{p}+\frac{1}{q}=1$. If $a, b \geq 0$, then prove
   $a^{1/p}\, b^{1/q} \leq \frac{a}{p}+\frac{b}{q}$
   Where the sign of inequality holds iff $a^p = b^q$

A P.N. 6 (Lemma-6)

2. State and prove Holder's inequality.

A P.N. 8 (Theorem-7)

3. State and prove Minkowski's inequality.
   $\lceil \|x\| - \|y\| \rceil \leq \|x-y\|$

A P.N.9 (Th.8)

4. Show that the linear spaces $R^n(Euclidean)$ and $C^n(Unitary)$ of n-tuples $x = (x_1, x_2, \dots \dots . x_n)$ of real and complex numbers are Banach space under the norm
   $\|x\| = \{\sum_{i=1}^{n} |x_i|^2\}^{1/2}$

A P.N. 16 (Exq.2)

5. Let p be a real number s.t. $1 \leq p \leq \infty$. Show that the space ${l_p}^n$ of all n-tuples of scalars with the norm defined by
   $\|x\|_p = \{\sum_{i=1}^{n} |x_i|^p\}^{1/p}$
   is a Banach space.

A P.N.18 (Exam.3)

6. Consider the linear space of all n-tuples $x = (x_1, x_2, \dots \dots . x_n)$ of scalars and define the norm by $\|x\|_\infty = max\{|x_1|, |x_2|, \dots |x_n|\} = l_\infty^n$ show that $(l_\infty^n, \|.\|_\infty)$ is a Banach space.

A P.N.20 (Exa.4)

7. Let (X) be a linear space of all bounded continuous scalar valued function defined on a topological space X. Then show that c(X) is a Banach space under the norm

$$\|f\| = sup\{|f(x)|: x \in X\}, \quad f \in C(X)$$

A P.N.21 (Exa.5)

8. If T be a linear transformation from a normed linear space N into the normed space N^1, then prove that the following statement are equivalent:
   (i) T is continuous
   (ii) T is continuous at the origin i.e. $x_n \to 0 \Longrightarrow T(x_n) \to 0$.
   (iii) T is bounded i.e. , $\exists$ $real\ k \ \geq 0\ s.t. \|T(x)\| \leq \|x\|$ for all $x \in N$
   (iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image T() is bounded set in N^1.

A P.N.26 (Th-1)

9. If T be a bounded linear transformation of normed space N into normed space N^1 then prove that the following statements are equivalent:
   (i) $\|T\| = sup\left\{\frac{\|T(x)\|}{\|x\|} : x \neq 0, x \in N\right\}$
   (ii) $\|T\| = inf.\{k : k \geq 0, \|T(x)\| \leq \|x\|, \forall\ x \in N\}$
   (iii) $\|T\| = sup\ \{\|T(x)\|: \|x\| \leq 1\ ,\ x \in N\}$
   (iv) $\|T\| = sup\ \{\|T(x)\|: \|x\| = 1,\ x \in N\}$

A P.N.28(Th.2)

10. If N, N^1 be normed linear spaces and B (N, N^1) is the set of all bounded (or continuous) linear transformation from $N\ into\ N^1$, then B (N, N^1) is also normed linear space under the norm
    $\|T\| = sup\ \{\|T(x)\|: \|x\| \leq 1, \forall\ x \in N\}$

    w.r.t. pointwise linear operations

    $(T + S0\ (x) = T\ (x) + S(x)\ and\ (XT)(x) = \alpha T(x)$ for real α. also

    B(N, N^1) is complete if N^1 is complete i.e. B(N, N^1) is a Banach space if N^1 is a Banach space.

A P.N.30 (Th.3)

11. Let $N\ and\ N^1$ be normed linear spaces over the same scalar field and let T be a linear transformation of $N\ into\ N^1$, then prove that T is bounded if it is continuous.

A. P.N. 36 (Th.8)

12. If N be a normed linear space, then show that the two norms $\| \quad \|_1 \| \quad \|_2$ defined in N are equivalent iff $\exists$ positive real numbers and b s.t.

$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1, \forall x \in N$

A. P.N. 38 (Th.10)

13. State and prove Reisz Lemma.

A. P.N. 41 (Th.14)

14. Let $X_1, X_2, \ldots . X_n, Y$ be normed linear spaces over the same field of scalars and let $f : X_1 \times \ldots . . \times X_n \rightarrow Y$ be a multilinear mapping, then show that f is continuous iff there exists a number m > 0 s.t.

    $\|f(x_1, x_2, \ldots . x_n\| \leq m\|x_1\|\|x_2\| \ldots . . \|x_n\|$

A. P.N. 46 (Th.1)

15. If B and B^1 be Banach spaces an T a continuous linear transformation of B onto B^1, then show that the image of every open centre at origin in B contains an open sphere centered at origin in B^1.

A. P.N. 48 (Lemma)

16. Let N and N^1 be normed linear spaces and D be a subspace of N, then show that a linear transformation $T : D \rightarrow N^1$ is closed iff its graph T_G is closed

A. P.N. 54 (Th.3)

17. State and prove the closed graph theorem.

A. P.N. 55 (Th.4)

18. State and prove the uniform boundedness theorem

A. P.N. 57(Th.5)

19. If f be a functional defined on a linear subspace M of a real normed linear space N, $x_0 \in M$ and

    $M_0 = [M \cup \{x_0\}] = \{x + \alpha x_0 : x \in M \text{ and } \alpha \text{ is real}\}$

    is the linear subspace spanned by M and x_0, then show that f can be extended to a functional & defined on M_0 .t. $\|f_0\| = \|f\|$

A. P.N. 62(Lemma)

20. If f be a functional defined on a linear subspace M of a complex normed linear space $N, x_0 \in M$ and

    $M_0 = [M \cup \{x_0\}] = \{x + \alpha\, x_0 : x \in M \text{ and } \alpha \text{ is real}\}$

    is the linear subspace spanned by M and x_0, then f can be extended to a functional f_0 defined on M_0 $s.t. \|f_0\| = \|f\|$

A. P.N. 62. 65-66

21. If N be a normed linear space and x_0 is a non-zero vector in N, then ∃ a continuous linear functional F defined on the conjugate space $N^*\ s.t.$

$f(x_0) = \|x_0\|\ and\ \|F\| = 1$ Prove it

A. P.N. 67(Th.2)

22. If M be a closed linear subspace of a normed linear space N and x_0 be a vector in N, but not in M with the property that the distance from x_0 to M i.e. $d\,(x_0, m) = d > 0$, then prove that ∃ a bounded linear functional $F \in N^*\ s.t. \|F\| = 1, F(x_0) = d\ and\ F(x) = 0\ \ \forall\, x\ \in M\ i.e. F(M) = \{0\}$

A. P.N. 69(Th.5)

23. Let N be an arbitrary normed linear space, then show that for each vector x in N induces a functional $F_x\ on\ N^{**}$defined by $F_x(f) = f(x)\ \forall\, f \in N^* s.t.$

$\|F_x\| = \|x\|$. Further the mapping $J : N \to N^{**}$ defined as $J(x) = F_x\ \forall\, x \in N$ is an isometric isomorphism of N into N^{**}.

A. P.N. 71 (Th.6)

24. Let X be a complex inner product space, then show the following results :

(i) $(\alpha x - \beta y, z) = \alpha(x, z) - \beta\,(y, z)$

(ii) $(x, \beta y + \gamma z) = \overline{\beta}(x, y) + \overline{\gamma}\,(x, z)$

(iii) $(x, \beta y - \gamma z) = \overline{\beta}(x, y) - \overline{\gamma}\,(x, z)$

(v) $(x, 0) = 0\ and\ (0, x) = 0\ \ \forall x \in X$

A. P.N. 77 (Th.1)

25. State and prove the Schwarz inequality

A. P.N. 78 (Th.2)

26. If X is an inner product space, then prove that

$\|x\| = (x, x)^{1/2}$ is a norm on X

A. P.N. 78 (Th.3)

27. If B is a complex Banach space whose norm obeys the parallelogram law, and if an inner product is defined on B by

$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$

Then B is a Hilbert space. Prove it.

A. P.N. 82 (Th.5)

28. A closed convex subset k of a Hilbert space H contains a unique vectors of smallest norm. Prove it.