Program : M.A./M.Sc. (Mathematics)
M.A./M.Sc. (Final)
Paper Code:MT-09
Integral Transforms and Integral Equations
Section – B
(Short Answers Questions)

1. Find Laplace Transform of $t^2.u(t-3)$, where $u(t-3)$ is a unit step function.
2. Evaluate Laplace transform of the function :
   $f(t) = \sin at - at\cos at + \frac{\sin t}{t}$
3. Find Laplace transform of the function $\sin\sqrt{t}$ and hence obtain the Laplace transform of $\frac{ws\ \sqrt{t}}{\sqrt{t}}$
4. If $f(t)$ continuous for all $t > 0$ and is of exponential order as $t \to \infty$ and if $f(t)$ is of class A, then show that :
   $\lim_{t\to\infty} f(t) = \lim_{p\to 0} P\ L\ [f(t);p]$
5. Prove that $L[U(t-a);p] = \frac{e^{-ap}}{p}$, where $U(t-a)$ is the Heaviside's unit step function.
6. Find $L\ [S_\in(t);p]$ where $S_\in(t)$ is dirac delta function and hence show that $\lim_{\in\to 0} L[S_\in(t);p] = 1$.
7. Prove that $L^{-1}\left[\frac{e^{-1/p}}{\sqrt{p}}\ ;t\right] = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$
8. Find $L^{-1}\left\{\frac{1}{(p^2+a^2)^{3/2}}\ ;t\right\}$, hence obtain the unverse Laplace transform of $\frac{1}{(p^2+2p+5)^{3/2}}$.
9. Find inverse Laplace transform of :
   $\log\left(1+\frac{1}{p^2}\right)\ or \log\left(\frac{p^2+1}{p^2}\right)$
10. Find $L^{-1}\left[e^{-a\sqrt{p}}\right]$.
11. Obtain inverse Laplace transform of $\frac{1}{(p+1)\ (p^2+1)}$ using convolution theorem.
12. Evaluate : $L^{-1}\left[log\left(\frac{p+\sqrt{p^2+1}}{2p}\right)\ ;t\right]$.
13. Solve $\frac{d^4y}{dx^4} - y = 1$, subject to conditions $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
14. Solve :
    $(2D^2+3D-2)\ y = 0, y(0) = 1, y(t) \to 0\ as\ t \to \infty$.
15. Find the solution of $\frac{\partial u}{\partial t} = 3\ \frac{\partial^2 u}{\partial x^2}$, given that $u_x(0,t) = 0, u\left(\frac{\pi}{2}\ ,t\right) = 0\ and\ u(x,0) = 30\cos 5x$.

16. Solve the boundary value problem:

$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ with the boundary conditions:

$u(x,0) = 0,\ u_t(x,0) = 0 \quad ;\ x > 0$

$u(0,t) = f(t),\ \lim_{x\to\infty} u(x,t) = 0 \quad ; t \geq 0$

17. Solve :

$ty'' + y' + 4ty = 0\ if\ y(0) = 3, y'(0) = 0$

18. Solve :

$(D^2 + 9)y = \cos 2t, if\ y(0) = 1, y\left(\frac{\pi}{2}\right) = 1.$

19. If $\phi(p)$ is the fourier sine transform of $f(t)$ for $p > 0$ then for $p < 0,\ F_3\{f(t); p\} = -\phi(-p)$, show it

20. Find the fourier transform of $f(t) = \begin{cases} 1, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$

21. Find the fourier cosine transform of e^{-t^2}.

22. Prove that :

$P^n f^{(m)}(p) = i^{m+n} \sum_{r=0}^{q} \frac{m!n!}{r!(n-r)!(m-r)!} \bar{F}\{t^{m-r}. f^{(n-r)}(t); P\}$

23. Solve the integral equation for $f(t)$:

$\int_0^\infty f(t) cps\ pt\ dt = \begin{cases} 1-p, & 0 \leq p \leq 1 \\ 0, & p > 1 \end{cases}$

Hence deduce that $\int_0^\infty \frac{sin^2 t}{t^2}\ dt = \frac{\pi}{2}$.

24. Find $f(t)$ if its sine transform is $\frac{e^{-ap}}{p}. Hence\ deduce\ F_S^{-1}\left(\frac{1}{p}\right)$.

25. Prove that:

$$M\{(1+x^a)^{-b}; p\} = \frac{\Gamma\left(\frac{p}{a}\right)\ \Gamma\left(b - \frac{p}{a}\right)}{a\,\Gamma(b)}\ ;\ 0 < Re\,(p) < Re\,(ab).$$

26. Prove that if n is a positive integer :

$M\left\{\left(x\frac{d}{dx}\right)^n f(x); p\right\} = (-1)^m P^n F(p)\ where\ M\{f(x)\} = F(p).$

27. If m is a positive integer and $\alpha \neq 0$ then prove that :

$M\left\{\left(\frac{d}{dx}x\right)^m f(x); p\right\} = (-1)^m\ F(p)\ \ where\ M\{f(x)\} = F(p).$

28. Derive the Mellin Inversion theorem.

29. Derive convolution theorem for Mellin transform.

30. If $M\{f(x)\} = F(p)$, then prove that :

$M\left\{x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx}; p\right\} = P^2 F(p)$

31. Find the Hankel transform of $f(x) = \begin{cases} 1, & 0 < x < a,\ v > 0 \\ 0, & x > a,\ v = 0 \end{cases}$

32. Find the Hankel transform of the function :

$f(x) = \begin{cases} a^2 - x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

taking $x\,J_0\,(px)$ as the kernel.

33. Find the Hankel transform of :

(i) $\frac{\cos ax}{x}$

(ii) $\frac{\sin a x}{x}$

taking $x\, J_0\,(px)$ as the kernel.

34. Find the Hankel transform of the function :

$$f(x) = \begin{cases} x^n, & 0 < x < a \\ 0, & x > a \end{cases} \qquad (n > -1)$$

taking $x\, J_n\,(px)$ as the kernel.

35. Find the Hankel transform of $e^v e^{-ax}$, taking $x\, J_v\,(px)$ as the kernel.

36. If $H_v\{f(x);p\} = \int_0^\infty f(x) J_v(px)(xp)^{\frac{1}{2}}dx,\ p > 0$, then show that:

$$H_v\left\{x^{v-\frac{1}{2}}\ e^{-ax}\ ;\ p\right\} = \frac{2^v \Gamma(v+\frac{1}{2})\ P^{v+\frac{1}{2}}}{\sqrt{\pi}\ (a^2+p^2)^{v+\frac{1}{2}}}$$

Where $Re\ (a) > \ 0\ and\ Re\ (v) > \frac{1}{2}$

37. Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, x\ > 0,\ t > 0$ subject to conditions :

(i) $U(o,t) = 0$

(ii) $U\ = \begin{cases} 1, & 0 < x < 1 \\ 0, & x\ \geq 1 \end{cases}$ when t = 0

(iii) $U\ (x,t)$ is bounded

38. The temperature $U\ (x,t)$ in the semi-infinite rod $0\ \leq x\ \leq\ \infty$ is determined by the D.E.

$$\frac{\partial U}{\partial t} = \ k\ \frac{\partial^2 U}{\partial x^2}$$

Subject to conditions :

(i) $U = 0\ when\ t\ = 0,\ \ x \geq 0$

(ii) $\frac{\partial U}{\partial x} = -\mu\ (a\ constant)\ \ when\ \ x\ = 0,\ t\ \geq 0$

Making use of cosine transform, show that

$$U(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos pu}{p^2}\left(1 - e^{-kp^2 t}\right)dp$$

39. Solve $\frac{\partial^4 U}{\partial x^4} + \frac{\partial^2 U}{\partial x^2} = 0, -\infty < x < \ \infty,\ y\ \geq 0$, satisfying the conditions :

(i) U and its partial derivatives tend to zero as $x \to \pm\infty$

(ii) $U = f(x), \frac{\partial U}{\partial y} = 0\ for\ y = 0$

40. Find the solution of the linear diffusion equation :

$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t}$ in a semi-infinite rod $x \geq 0$, satisfying the boundary conditions

(i) $U\ (0,t) = \ f(t),\ \ t\ \geq 0$

(ii) $U\ (x,t) \to 0\ as\ x \to \infty$

and he initial conditions $U\ (x,0) = 0$

41. Apply Hankel transform (of zero order) to solve the differential equation:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad 0\ \leq r \leq \infty, z \geq 0$$

satisfying the following conditions :

(i) $U\ \to 0\ as\ z\ \to \infty\ and\ r\ \to \infty$

(ii) $U = f(r) on\ z = 0, r\ \geq 0$. It is given that $U(r,z)$ is bounded.

42. The magnetic potential U for a circular disc of radius a and strength w, magnetized parallel to its axis. Satisfying Laplace's equation is equal to $2\pi w$ on the disc itself and vanishes at exterior point in the plane of the disc. Show that at the point $(r, z), z > 0$

$$U = 2\pi w \int_0^{\infty} e^{-pz} J_0(pr) J_1(ap) dp$$

43. Show that the function $g(x) = (1 + x^2)^{-\frac{3}{2}}$ is a solution of the volterra integral equation:

$$g(x) = \frac{1}{1+x^2} - \int_0^x \frac{1}{(1 + x^2)} g(t)dt$$

44. Show that the function $g(x) = 1$ is a solution of the Fredholm integral equation.

$$g(x) + \int_0^1 x\,(e^{xt} - 1) g(t)dt = e^x - x$$

45. Form a integral equation corresponding to the differential equation :

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

With initial conditions : $y(0) = 1 = y'(0) = 0$

46. Convert the following differential equation into an integrall equation :

$$\frac{d^2y}{dx^2} + \lambda xy = f(x)\,; y(0) = 1, y'(0) = 0$$

47. Transform $\frac{d^2y}{dx^2} + xy = 1\,;\; y(0) = 0, y(1) = 1$ into an integral equation.

48. Solve the homogeneous fredholm integral equation:

$$\phi(x) = \lambda \int_0^1 e^{x+t}\, \phi(t)dt$$

49. Solve the homogeneous fredholm integral equation of the second kind.

$$g(x) = \lambda \int_0^{2\pi} \sin(x + t)\; g(t)dt$$

50. Find eigen values and eigen functions of the homogeneous integral equation.

$$g(x) = \lambda \int_0^1 k(x, t)\; g(t)dt$$

51. Solve :

$$g(x) = e^x + \lambda \int_0^1 2e^x\, e^t\, g(t)dt$$

52. Solve the integral :

$$g(x) = f(x) + \lambda \int_{-1}^1 (xt + x^2t^2\; g(t)dt$$

Also find its resolvent kernel.

53. Solve the following equation and find its eienvalues :

$$g(x) = + + \lambda \int_0^{\pi/2} \cos\,(x - t)\; g(t)dt$$

54. Solve the integral equation :

$$g(x) = 1 \int_0^x \sin\,(x - t)\; g(t)dt$$

and verify your answer.

55. Solve for $f(x)$ the integral equation "

$$\int_0^{\infty} f(x)\sin px\,dx = \begin{cases} 1, & 0 \le p \le 1 \\ 2, & 1 \le p \le 2 \\ 0, & p > 2 \end{cases}$$

56. Solve :

$$\int_0^{\infty} f(x)\cos px\,dx = e^{-p}$$

57. Solve the following integral equation by the method of successive approximation :-

$$g(x) = \left(e^x - \frac{1}{2}e + \frac{1}{2}\right) + \frac{1}{2}\int_0^1 g(t)dt$$

58. Using iterative method, solve :

$$g(x) = f(x) + \lambda \int_0^1 e^{x-t}\, g(t)dt$$

59. By means of resolvent kernel find the solution of :

$$g(x) = e^x \sin x + \int_0^x \frac{2 + \cos x}{2 + \cos t}\, g(t)dt$$

60. By means of resolvent kernel, find the solution of:

$$g(x) = 1 + x^2 \int_0^{\pi} \frac{1 + x^2}{1 + t^2}\, g(t)dt$$

61. Using the method of successive approximation solve the integral equation :

$$g(x) = 1 + \int_0^x (x - t)g(t)dt \;\; taking\ g_0(x) = 0$$

62. Solve :

$$g(x) = x.2^x - \int_0^x 2^{x-t} g(t)dt\,, \quad g_0(x) = x.2^x$$

By using the method if successive approximation.

63. Prove that the eigen values of a symmetric kernel are real.
64. Show that the eigen functions of a symmetric kernel corresponding to distinct eigen values are orthogonal.
65. Show that if the sequence $\{g_k(x)\}$ be all the eigen functions of a symmetric L_2 −kernel with $\{\lambda_k\}$ as the corresponding eigen values. Then the series : $\sum_{n=1}^{\infty} \frac{|g_n(x)|^2}{\lambda^2 n}$ converges and its sum is bounded by C_1^2 which is an upper bound of the integral:

$$\int_a^b |k^2(x,t)|\, dt$$

66. Show that if the sequence $\{g_n(x)\}$ be all the eigen functions of a symmetric kernel k(x, t) with $\{\lambda_n\}$ as the corresponding eigen values then the truncated kernel $K^{(n+1)}(,t) = k\,(x,t) - \sum_{m=1}^{n} \frac{g_m(x)\bar{g}_m(t)}{\lambda_m}$ has the eigen values $\lambda_{n+1}, \lambda_{n+2}, \dots$ ...to which corresponds the eigen functions $g_{n+1}(x), g_{n+2}(x) \dots \dots .. The\ Kernel\ K^{n+1}(x,t)$ has no other eigenvalues or eigen functions.
67. Using Hilbert Schmidt theorem, find the solution of the symmetric integral equation

$$g(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1} (xt + x^2t^2)g(t)dt$$

68. Determine the eigenvalues and the corresponding eigen functions of the equation:

$$g(x) = f(x) + \lambda \int_0^{2\pi} sin(x+t)g(t)dt$$

Where $f(x) = x$ obtain the solution of this equation when λ is not an eigenvalue.

69. Using the recurrence relation find the resolvent kernel of the following kernel :

$k(x,t) = \sin x, \quad 0 \le x \le \pi$

70. Show by using fredholm's theory that the resolvent kernel for the integral equation with kernel $k(x,t) = 1 - 3xt$ in interval (0,1) is :

$$R(x,t\,;\,\lambda) = \left[\frac{4}{4-\lambda^2}\right]\left[1 + \lambda - \frac{(x-t)}{2} - 3\,(1-\lambda)xt\right], \lambda \neq 2$$

71. Using fredholm theory solve,

$$g(x) = e^x + \lambda \int_0^1 xt\, g(t)dt$$

72. Solve the following integral equation :

$$g(x) = x + \lambda \int_0^1 [4xt + x^2]g(t)dt$$

73. Using fredholm theory, solve :

$$g(x) = \cos 2x \int_0^{2\pi} \sin x \cos t \; g\,(t)dt$$

74. For the integral equation :

$$g(x) = f(x) + \lambda \int_a^b k\,(x,t)g(t)dt$$

Find $D(\lambda)$ *and* $D(x,t\,;\,\lambda)$ for the kernel:

$k(x,t) = \sin x\,; \;\; a = 0, \;\; b = \pi$