Program : M.A./M.Sc. (Mathematics)

M.A./M.Sc. (Final)

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Analysis and Advanced Calculus

Section – B

(Short Answers Questions)

1. Define Norm and write the set of axions of Normal linear space.

A P.N. 2

2. Write the Summability for a series $\sum f_n$ of functions in a Normal linear space.

A P.N. 3

3. If N be a Normed linear space and x, y, $\in N$then Prove $\lceil \|x\| - \|y\| \rceil \leq \|x - y\|$

A P.N.4

4. Show that every normed linear space is a metric space.

A P.N. 4

5. If N be a normed linear space with the norm $\|.\|$, then prove that mapping $f: N \rightarrow R$ s.t. $f(x) = \|x\|$ is continuous.

A P.N.5

6. Show that every convergent sequence in a normal linear space is a Cauchy sequence.

A P.N.5

7. Show that the limit of a convergent sequence is unique.

A P.N.6

8. Write the Reflexive, Symmetric and Transitive relations for factor (quotient) spaces.

A P.N.11

9. Show that the linear spaces R(real) and (Complex) are normed linear spaces under the norm $\|x\| = |x|, x \in R \; or \; C$ as the case may be.

   Also show that these spaces are complete and hence Banach spaces.

A P.N.15

10. If T be a linear transformation of a normed linear space N into normed linear space N^1. then prove that its inverse of T i.e. T^{-1} exists and is continuous on its domain of definition iff $\exists$ a constant $K \geq 0\, s.t.\, K\|x\| \leq \|T(x)\|\ V\ x \in N.$

A P.N.33

11. If T be a linear transformation from a normed linear space into normed space N^1, then show that T is continuous either at every point or at no point of N.

A P.N.34

12. If M be a closed linear subspace of a normed linear space N and T be a natural mapping (homomorphism) of N onto $\frac{N}{M}$ s.t. $T(x) = x + M$, then show that T is continuous linear transformation with $\|T\| \leq 1$.

A P.N.34

13. Show that the weak limit of a sequence is unique.

A P.N.37

14. Show that on a finite dimensional linear space X, all norms are equivalent.

A P.N.39-40

15. Prove that every compact subset of a normal linear space is complete.

A P.N.40

16. Prove that every compact subset of a normal space is bounded but the converse is not true.

A P.N.41

17. Let n be a normed linear space and suppose the set $S = \{x \in N: \|x\| = 1\}$ is compact then prove that N is finite dimensional.

A P.N.42

18. Let B and B^1 be Banach spaces. If T is a continuous linear transformation of B and B^1, then prove that T is an open mapping.

A P.N.52

19. Let N be a real normed linear space and suppose $f(x) = 0\ \forall\ f \in N^*$. Show that x = 0.

A P.N.68

20. If M be a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then prove that ∃ a functional F in conjugate space N^* s.t. $f(M) = \{0\} and\ f(x_0) \neq 0$.

A P.N.69

21. Show that the space ${l_2}^n$ consisting of all n types $x = (x_1, \dots \dots . x_n)$ of complex numbers and the inner product on ${l_2}^n$ is defined as $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$, where $y = (y_1, \dots \dots \dots y_n)$ is an inner product space.

A P.N.75

22. Show that the linear space l_2 consisting of all complex sequences $x = (x_n) s.t. \sum_{n=1}^{\infty} |x_n|^2$ is convergent.

A P.N.76

23. Show that the inner product in a Hilbert space is jointly continuous i.e. if $x_n \to x$ and $y_n \to y$, then $(x_n, y_n) \to (x, y)$ as $n \to \infty$

A P.N.80

24. If x and y are any two vectors in a Hilbert space H, then prove

$$\|(x + y)\|^2 + \|(x - y)\|^2 = 2(\|x\|^2 + \|y\|^2)$$

A P.N.81

25. If x, y are any two vectors in a Hilbert space H, then prove that

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

A P.N.81

26. Write the Pythagorean theorem statement and proof also.

A P.N.92

27. If S is a non empty subset of a Hilbert space H, then prove that S^1 is a closed linear subspace of H and hence a Hilbert space.

A P.N.94

28. Let M be a linear subspace of Hilbert space H. then prove that M is closed if and only if $M = M^{11}$

A P.N.96

29. If M is a closed linear subspace of a Hilbert space H. then show that $H = M + M^1$

A P.N.97

30. If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H, then show that the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

A P.N.100

31. Show that an orthonormal set S in a Hilbert space H is complete iff $x \perp S \Rightarrow x = 0 \ \forall x \in H$.

A P.N.104

32. Show that in the Hilbert space $l_2{}^n$, the set $\{e_1, e_2, \dots\dots e_n, \dots\dots\}$, where e_n is a sequence with 1 in the n^{th} place and 0's elsewhere is a complete orthonormal set.

A P.N.104

33. If an operator T on H is self-adjoint, then show that $(T_x, y) = (y, T_y) \forall x, y \in H$ and conversely.

A P.N.121

34. If T be a self-adjoint operator, then show that $T + T^*$ and T^*T are self-adjoint.

A P.N.122

35. If T is an arbitrary operator on Hilbert space H, then prove that T=0 iff $(T_x, y) = 0 \ \forall x, y \in H$.

A P.N.122

36. Show that an operator T on a complex Hilbert space H is self=adjoint iff (T_x, x) is real for all x.

A P.N.123

37. Let A be the set of all self-adjoint operators in $\beta(H)$, then show that A is a closed linear subspace of $\beta(H)$ and therefore A is a real Banach space containing the identity transformation.

A P.N.124

38. Write the conditions so that :

(a) The identity operator I and the zero operator 0 are positive operators.

(b) For any arbitrary operators T on H, TT^* *and* T^*T *are* positive operators.

A P.N.125

39. Prove that an operator T on a Hilbert space H is normal iff

$\|T^*x\| = \|T_x\| \quad \forall x \in H$

A P.N.126

40. If T is a Normal Operator on H, then prove that $\|T^2\| = \|T\|^2$

A126

41. If T is an operator on a Hilbert space H, then show that T is normal iff its real and imaginary parts commute.

A P.N.128

42. If P is a projection on a Hilbert space H, then prove that :
(a) $\|P_x\| \leq \|x\| \quad \forall x \in H$
(b) $\|P\| \leq I$

A P.N.138

43. If P is a projection on a Hilbert space H, then prove that:
(a) P is a positive operator.
(b) $0 \leq P \leq I$

A P.N.138

44. Show that a closd linear subspace M of a Hilbert space H is invariant under an operator $T \Leftrightarrow M^{\perp}$ is invariant under T^*

A P.N.139

45. Show that a closed linear subspace M of a Hilbert space H reduces an operator $T \Leftrightarrow M$ is invariant under both T and T^*

A P.N.139

46. If P and Q are projections on closed linear subspace M and N of a Hilbert space H, the prove that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$

A P.N.140

47. If x is an eigenvector of T corresponding to eigenvalue λ, and α is a non-zero scalar, then prove that α is also an eigenvector of T corresponding to same eigenvalue.

A P.N.141

48. If x is an eigenvector of T, then show that x cannot corresponding more than one eigenvalue of T.

A P.N.141

49. If T is a normal operator on a Hilbert space H, then prove that x is an eigenvector of T with eigenvalue λ iff x is an eigenvector of T^* with $\bar{\lambda}$ as eigenvalue.

A P.N.142

50. If T is a normal operator on a Hilbert space H then prove that each eigenspace of T reduces T.

A P.N.143

51. If T is normal operator on a Hilbert space H, then prove that eigenspaces of T are pairwise orthogonal.

A P.N.143

52. Prove that an operator T on a finite-dimensional Hilbert space H is singular $\Leftrightarrow$ there exists a non-zero vector x in H s.t. $T_x = 0$

A P.N.144

53. If T is an arbitrary operator on a finite dimentional Hilbert space H, then prove that the eigenvalues of T constitute a non empty finite subset of the complex plane. Furthermore, the number of points in this does not exceed the dimension n of the space H.

A P.N.144

54. Let X and y be any two Banach spaces over the same field K, then show the set of all functions tangential to a function f at v $\in V$, there is at most one function $\phi\ (x) = f(v) + g(x - v)$, where $g: x \rightarrow y$ is linear, where V is an non-empty open subset of X.

A P.N.152

55. Let X and Y be Banach spaces and V be the non-empty open subset of X. If $f: V \rightarrow Y$ and g: $V \rightarrow Y$ be differentiable in V and be any scalar in K, then show that the function $(f + g): V \rightarrow Y$ $and\ \alpha f: V \rightarrow Y\ defined\ by$ $a\, f(x) = a\, f(x)$, $(f + g)(x) = f(x0 + g(x)$ are differentiable in V and for all $v \rightarrow V$, $D(af)(v) = aDF(v)$, $D(f + g)(v) = Df(v) + Dg(v)$

A P.N.153

56. If X and Y be Banach spaces over the same field K of scalars and V be an open subset of X. Let $f : V \rightarrow Y$ is differentiable at $x \in V$, then show that all the directional derivatives of f exists at x and $D_v f(x) = D\, f(x).v$, where $v \in V$ is a unit vector.

A P.N.157

57. Let X be banach space over the field K of scalars and V be an open subset of X. If $f : V \rightarrow R$ be a function. Let u and v be any two distinct points in V s.t. $[u, v] \subset V$ and f is differentiable at all points of [u,v], then show that

$$f(v) - f(u) = Df\big(u + t(v - u)\big).(v - u) \quad where\ t \in (0,1)$$

A P.N.161

58. Let X and Y be any two Banach spaces over the same field K of scalars and V be an open subset of X. Let $f : V \to Y$ be a continuous function and let u and v be any two distinct points in V s.t. $[u, v] \subset V$ and f is differentiable in [u,v]. If $g : X \to Y$ be any continuous linear function, then show that $\|f(v) - f(u) - g(v - u)\| \leq C \|v - u\|$

A P.N.162

59. How that C^n map is continuous

A P.N.168

60. Let X and Y be Banach spaces over the same field k of scalars and V be an open subset of X. If $f : V \to Y$ be an n-times differentiable function on V, then prove for each permutation p of n and each point $(x_1, x_2, \dots . x_n) \in x^n$ and each $v \in V$,

$$D^n f(v)\left(x_{p(1)}, x_{p(2)}, \dots . x_{p(n)}\right) = D^n f(v)(x_1, x_2, \dots . x_n)$$

A P.N.171

61. Let X be a Banach space over the field k of scalars, and let I be an open interval in R containing [0,1]. If $\Psi: I \to X$ is (n+1) times continuously differentiable function of a single variable $t \in I$ then show that

$$\Psi(1) = \Psi(0) + \Psi^1(0) + \frac{\Psi^n(0)}{2!} \dots . + \frac{\Psi^n(0)}{n!} + \int_0^1 \frac{(1-t)^n}{n!} \; {\Psi_{(t)}}^{n+1} dt.$$

A P.N.173

62. Let [a,b] be a compact interval of R and let X be a Banach space over k, then prove that the set $S([a, b], X)$ of all step functions on [a, b] into X is a vector subspace of the bnach space B([a, b],X) into X.

A P.N.185

63. Let f be a regulated function on a compact interval [a, b] of R into a Banach space X over k, and c be any point of [a, b], then show that the restriction of f to [a,c] {respectively [c, b]) is a regulated function on [a, c] (respectively [c, b]) into X and $\int_a^b f = \int_a^c f + \int_c^b f$ is.

A P.N.188

64. Let f be a regulated function on a compact interval [a, b] of R into a Banach space X, then show that each $t \in (a, b)$, the function $F: [a, b] \to X, F(t) = \int_a^t f, t \in [a, b]$ is continuous.

A P.N.189

65. Let f be a continuous function on a compact interval [a, b] of R into a Banach space X over K. Let F be he function $t \to \int_a^t f \; on \; [a, b]$ into X.

Let g be any differentiable function on [a, b] into X s.y. Dg=f, then prove that F is differentiable, DF = f and

$\int_a^b f + F(b) - F(a) = g(b) - g(a)$

A P.N.190

66. Let f be a C^1 map on a compact interval [a, b] into a compact interval [c, d] of R and let g be a continuous function on [c, d] into a Banach space x over k, then prove that

$\int_a^b (D\, f\,(s)\, g\,(f\,(s))ds = \int_{f(a)}^{f(b)} g(t)dt$

A P.N.190

67. Let f be a regulated function on a compact interval [a, b] of R into R s.t. a < b and for all t in [a, b], $f(t) \geq 0$, then show $\int_a^b f(t)dt \geq 0$. If f is continuous at a point c of [a, b] and $f(c) > 0$ then also show $\int_a^b f(t)dt > 0$

A P.N.192

68. Let f be a continuous function on a compact interval [a, b] of R nto the topogical dual X^* of a Banach space X over R s.t. a < b and for each C^1map g on [a, b] into X with $g(a) = g(b) = 0$ for each $t \in [a, b]$

A P.N.193

69. Let I be an open integral of R, let W be an open subset of a Banach space X over K let (t_0, x_0) be point of $I \times W$ and Let g be a continuous map of $I \times W$ into X, then prove that a continuous map $h : I \to W$ is an integral solution for g at (t_0, x_o) iff for each $t \in I$.

$h(t) = x_0 + \int_{t_0}^{t} g\,(s, h\,(s))ds$

A P.N.197

70. Let u be a non-negative continuous function on an interval {0, c}, (C > 0) satisfying the inequality $u(t) \leq at + k \int_u^t u(s)ds$ for all $t \in [o, c]$. Then show that $u(t) \leq \frac{a}{k}(e^{kk} - 1)$ for $t \in [0, c]$

A P.N.198

71. Write the statement and proof of the Global uniqueness theorem.

A P.N.202

72. Write maximal integral solution for g at (t_0, x_0)

A P.N.205