Program : M.A./M.Sc. (Mathematics)
M.A./M.Sc. (Previous)
Paper Code:MT-03
Differential Equations, Calculus of Variations & Special Functions
Section – B
(Short Answers Questions)

1. Show that the differential equation.
   $$y + 3x\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right)^3 + \left(x^2 + 2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 0$$
   Is an exact equation and find its first integral.

A. MT-03, P. 3

2. Find the general solution of the Riccati's equation.
   $$\frac{dy}{dx} = 2 - 2y + y^2$$
   Whose one particular solution is $(1 + tan\, x)$

A. MT-03, P. 11

3. Solve :
   $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4\left(\frac{dy}{dx}\right)^2 = 0$$

A. MT-03, P. 17

4. Solve :
   $$sin^3y\frac{d^2y}{dx^2} = \cos y$$

A. MT-03, P. 14

5. Solve :
   $$nx^3\frac{d^2y}{dx^2} = \left(y - x\frac{dy}{dx}\right)^2$$

A. MT-03, P. 21

6. Solve :
   $$yz + xyz)\,dx + (zx + xy + xyz) + (xy + xyz)dx = 0$$

A. MT-03, P. 30

7. Solve :
   $$z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - zx)dz = 0$$

A. MT-03, P. 33

8. Solve :
   $$(e^xy + e^z)dx + (e^yz + e^x)dy + (e^y - e^xy - e^yz)dz = 0$$

A. MT-03, P.38

9. Solve :
$(xdx + ydy + zdz)^2 z = \{(z^2x^2y^2)(xdx + ydy + zdz)dx\}$

A. MT-03, P. 43

10. Solve :
$s - t = \frac{x}{y^2}$

A. MT-03, P. 48

11. Solve (By Monge's method) :
$(x - y)(xr - xs - ys - yt) = (x + y)(p - q)$

A. MT-03, P. 55

12. Solve :
$sr + 6s + 3t + 2(rt + s^2) + 3 = 0$

A. MT-03, P. 62

13. Solve :
Show that a surface passing through the circle $z = 0, x^2 + y^2 = 1$ and satisfying the differential equation $s = 8xy \; is \; z = (x^2 + y^2)^2 - 1$

A. MT-03, P. 49

14. Solve the following P.D.E.:
$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}, \quad 0 < x < \pi, \;\; y > 0$
Satisfy the boundary conditions :
(i) $Z = 0 \; where \; x = 0$
(ii) $Z = 0 \; where \; x = \pi$
(iii) $Z = sin\, 3x \; where \; y = 0$

A. MT-03 , P. 74

15. Reduce $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

A. MT-03, P. 96

16. Solve the harmonic equation:
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Satisfying the conditions:
$U(x, 0) = 0, U(x, a) = \sin\frac{\pi x}{l}$
$U(0, y) = U(t, y) = 0$

A. MT-03, P. 88

17. By separation of variables show that one dimensional wave equation:
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$
Has solution of the form A $\exp(tinx \pm inct)$ where A and n are constant

A. MT-03, P. 86

18. A tightly stretched string which has fixed end points $x = 0\ and\ x = l$ is initially in a position given by $y = ksin^3\left(\frac{\pi x}{e}\right)$. It is released from rest from this position. Find the displacement $y(x, t)$.

A. MT-03, P. 87

19. Reduce the equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

To canonical form and hence solve it.

A. MT-03, P.94

20. Find the eigen values λ's and corresponding eigen functions $y_n(x)$ for the equation $y'' + dy' = 0$ under the boundary condition $y(0) = 0\ and\ y\left(\frac{\pi}{2}\right) = 0$

A. MT-03, P. 104

21. Find eigen values and eigen functions for the following boundary solve problem.
$y'' + 2y' + \lambda y = 0\ \ ;\ y(0) = 0,\ \ y(\pi) = 0$

A. MT-03, P. 106

22. Find the eigen values and eigen functions for the boundary value problem.
$y'' - 3y' + 2(1 + \lambda)y = 0, \qquad y(0) = 0,\ \ y(1) = 0$

A. MT-03, P. 108

23. Solve the following strem Liouville problem.
$y'' + \lambda y = 0,\ \ y'(-\pi) = 0,\ \ y'(\pi) = 0$

A. MT-03, P. 111

24. Find the externals of the functional:

$$I(y, z) = \int_0^{\pi/2} [y^2 + z^2 + 2yz]\ dt$$

With the boundary condition $y(0) = 0, y\left(-\frac{\pi}{2}\right) = -1, z(0) = 0, x\left(\frac{\pi}{2}\right) = 1$

A. MT-03, P. 133

25. Test for an external of the functional:

$$F[y(x)] = \int_0^{\pi/2} (x^2 - y^2)dx,\ \ y(0) = 0, y\left(\frac{\pi}{2}\right) = 1$$

A. MT-03, P. 135

26. Find the curve with fixed boundary revolves such that its rotation about x-axis generated minimal surface area.

A. MT-03, P.131

27. Obtain the surface of minimum area, stretched over a given closed curve C, enclosing the domain D in the $x - y$ plane.

A. MT-03, P. 149

28. Solve in series $(2 - x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$

A. MY-03, P. 161

29. Solve the legendre's equation:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+x(x+1)y=0$$

A. MT-03, P. 162

30. Solve in series : $2x^2\,\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(1-x^2)y=0$

A. MT-03, P. 164

31. Solve the Gauss hupergeometric equation:

$$x(1-x)\frac{d^2y}{dx^2}+\;[(\gamma-(1+\alpha+\beta)x]\frac{dy}{dx}-\alpha\beta y=0$$

In series in the neighbourhood of the regular singular point $x=\infty$

A. MT-03, P. 150, 166

32. Solve in series :

$$x(1-x)\frac{d^2y}{dx^2}+(1+5x)\frac{dy}{dx}-4y=0$$

A. MT-03, P. 170

33. Solve :

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0$ in series

A. MT-03, P. 173

34. Solve in series :

$x^2\frac{d^2y}{dx^2}+(x+x^2)\frac{dy}{dx}+(x-9)y=0$ in series

A. MT-03, P. 176

35. Integrate in descending series the Legendre's equation.

A. MT-03, P. 177

36. If $|z|<1$ *and if* $Re(x)>Re(b)>0$ *then*

$$2f_1(a,b\,;c\,;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{e-b-1}(1-zT)^{-a}dt$$

A. MT-03, P. 186

37. If $Re\ Cc-a-b)>0\,, Re\ (c)>\ 0\ then$

$$2f_1(a,b\,;c\,;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

A. MT-03, P. 186

38. Prove (Kummer's Theorem)

$$2f_1(a,b\,;1-a+b\,;-1)=\frac{\Gamma(1-a+b)\Gamma(1+\frac{b}{2})}{\Gamma(1+b)\Gamma(1+\frac{b}{2}-a)}$$

A. MT-03, P. 187

39. Prove that :

$$2F_1\left[\frac{a}{2},\frac{a}{2}+\frac{1}{2};z^2\right]=\frac{1}{2}\,[(1-z)^{-a}+(1+z)^{-a}]$$

A. MT-03, P. 190

40. Prove that :

$$B(\lambda, c-\lambda)2F_1(a,b\,;c\,;z) = \int_0^1 t^{\lambda-1}\,(1-t)^{c-\lambda-1}\,2f_1\,(a,b\,;\,\lambda;\,zt)dt$$

Where $|z| < 1, \lambda > 0, c - \lambda < 0$

A. MT-03, P. 191

41. Show that:

$\Gamma(a)\Gamma(b)2f_1\left(a,b\,;\frac{1}{2};z\right) = \int_0^\infty \int_0^\infty e^{-u-v}\cosh\{2\sqrt{(uv)}\,z\}u^{a-1}v^{b-1}dudv$

Provided $Re\,(a) > 0$ and $Re\,(b) > 0$

A. MT-03, P. 194

42. Show that :

$\lim_{c\to -n}\frac{1}{\Gamma(c)}\,2f_1(a,b\,;c\,;z) = \frac{(a)_{n+1}(b)_{n+1}}{\llcorner(n+1)}z^{n+1}$

$2f_1(a+n+1, b+n+1; n+2\,;z)$

A. MT-03, P. 195

43. Prove that :

$$2f_1(a,b\,;c\,;z) = \frac{1}{B(b,c-b)}\int_0^1 u^{b-1}\,(1-u)^{c-b-1}\,(1-zu)^{-a}\,du$$

Where c>v>0 hence prove that $2f_1(1,2\,;3\,;z) = \log\{e(1-z)^{1/2}\}^{-2/z}$

A. MT-03, P. 196

44. Show that if $Re\,(b) > 0$ and if n is a non negative integer then :

$$2f_1\left(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; b+\frac{1}{2}; 1\right) = \frac{2^n(b)_n}{(2b)_n}$$

A. MT-03, P. 197

45. Show that :

$\int_0^t x^{\frac{1}{2}}\,(1-x)^{-\frac{1}{2}}\,[1-x^2(t-x)^2]^{-\frac{1}{2}} = \frac{1}{2}\pi t\,2f_1\left[\frac{1}{4},\frac{3}{4};1\,;\frac{t^4}{16}\right]$

A. MT-03, P. 198

46. It is the complete elliptic integral of first kind being:

$$L = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\,sin^2\phi}}$$

Show that:

$K = \frac{\pi}{2}\,2f_1(\frac{1}{2},\frac{1}{2}\,;1\,;\,k^2)$

A. MT-03, P. 196

47. Show that :

$\int_0^t x^{\frac{1}{2}}\,(1-x)^{-\frac{1}{2}}\,[1-x^2(t-x)^2]^{-\frac{1}{2}}\,dx = \frac{1}{2}\pi t\,2f_1\left[\frac{1}{4},\frac{3}{4};1\,;\frac{t^4}{16}\right]$

A. MT-03, P. 198

48. If $|z| < 1$ *and* $\left|\frac{z}{1-z}\right| < 1$ *then*:

$$2f_1(a,b\,;c\,;z) = (1-z)^{c-a-b}\;\;2f_1(c-a,c-b\,;c;z)$$

A. MT-03, P. 202

49. If $|z| < 1$ *and* $\left|\frac{z}{1-z}\right| < 1$ *then*:

$$2f_1(a,b\,;c\,;z) = (1-z)^{-a} \quad 2f_1(a,c-b,\ ;c;\tfrac{z}{z-1})$$

A. MT-03, P. 202

50. If $|z| < 1$ and $\left|\frac{z}{1-z}\right| < 1$ $then$:

$$2f_1(a,b\,;c\,;z) = (1-z)^{-b} \quad 2f_1(c-a,b\,;c\,;\frac{z}{z-1})$$

A. MT-03, P.202

51. Show that :

$$\frac{d}{dx}\,[2f_1\,(a,b;c;x)] = \frac{ab}{c}\quad 2f_1\,(a+1,b+1\,;c+1,x)$$

A. MT-03, P. 204

52. If m is a positive integer and $|x| > 1$ show that:

$$2f_1\left(m+\frac{1}{2},m+\frac{2}{2};1;\frac{1}{x^2}\right) = \frac{(-1)^m x^{m+1}}{\llcorner m}\frac{d^m}{dx^m}\left\{\frac{1}{\sqrt{x^2+1}}\right\}$$

A. MT-03, P. 213

53. Prove that :

$$\frac{d^m}{dx^m}\,[x^{\alpha+1+m}\ \ 2f_1\,(a,b\,;c;x)] = (a)_m x^{n-1}\ \ 2f_1\,(a+m,b\,;c;x)$$

A. MT-03, P. 214

54. Prove that if $(a+b+c) > 0$ then

$$\lim_{x\to 2}\{(1-x)^{a+b+c}\ \ 2f_1(a,b\,;c;x)\} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

A. MT-03, P. 214

55. Prove that:

$$1f_1(a,b;z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-t}\ \ t^{\alpha-1}\ \ 0f_1\,(-\,;b\,;zt)$$

A. MT-03, P. 215

56. If c is neither zero nor a negative then:

$$1f_1(a;c;z) = e^z\ 1f_1\,(c-a;c\,;-z)$$

A. MT-03, P. 212

57. Show that :

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h_1^n\,P_n(x), \qquad |x| \le 1, \quad |h| < 1$$

A. MT-03, P. 222

58. Show that :

$$P_n(x) = \frac{1}{2^n \llcorner n}\frac{d^n}{dx^n}(x^2-1)^n$$

A. MT-03, P. 223

59. Show that :

$$(2n+1)x\ \ P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x)$$

A. MT-03, P. 226

60. Show that:

$$n\,P_n(x) = x\,P_n^1\,(x) - P_{n-1}^1\,(x)$$

A. MT-03, P. 227

61. Prove that :