Program : M.A./M.Sc. (Mathematics)
M.A./M.Sc. (Previous)
Paper Code:MT-02
Real Analysis and Topology
Section – C
(Long Answers Questions)

1. If \$ is a semiring, then the following statements hold :
   (i) If $A \in \$ \; and \; A_1, A_2, \dots\dots, A_n \in \$ \; then \; A - \bigcup_{i=1}^{n} A_i \; is$ finite union of pairwise disjoint sets of \$.
   (ii) For every $\{A_n\}$ of pairwise disjoint members of \$, the set $A = \bigcup_{i=1}^{n} A_n$ is a σ-set.
   (iii) Countable union and finite intersection of σ-set are σ-sets.

A. (P. 08)

2. Let \$ be an algebra of sets of a set X and $\{A_n\}$ be a sequence of sets in \$. Then $\exists$ a sequence $\{B_n\}$ of sets in \$ such that $B_i \cap B_j = \emptyset \; if \; i \neq j \; and \; \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

A. (P. 11)

3. The outer measure of an interval is its length.

A. (P. 21)

4. The union of two measurable sets is a measurable set.

A. (P. 29)

5. Show that every interval is measurable.

A. (P. 35)

6. There exists a non-measurable set in the interval [0, 1].

A. (P. 41)

7. Let f and g be measurable functions defined on a measurable set E, and C be a constant. Then the functions $f \pm c, cf, -f, f \pm g, |f|, f^2, fg$ are measurable. Further if $g(x) \neq 0, \; \forall x \; \in E \; then \frac{1}{g} \; and \frac{f}{g}$ are also measurable.

A. (P. 46)

8. Let f be a measurable function, finite almost every where defined on the closed interval E = [a, b]. The for all $\sigma > 0 \; and \; \in > 0$ there exists a continuous function $\emptyset$ defined on E such that :
$$m\left(\{x \in E : |f(x) - \emptyset(x)| >, \sigma\}\right) < \in$$

A. (P. 57)

9. Let E be measurable set, with $m(E) < \infty \; and < f_n >$ a sequence of measurable functions defined on E. Let f be a real valued measurable function such that for each $x \in E, f_n(x) \to f(x)$. Then for given $\in < 0 \; and \; \delta > 0$, there is measurable set $A \subset E$, with $m(A) < \delta$ and an integer n_0 such that for all $x \in E - A$ and all $n \geq n_0 \quad |f_n(x) - f(x)| < \in$.

A. (P. 62)

10. Let $< f_n >$ be a sequence of measurable functions defined on a measurable set E, that converge point wise to a function f defined on E. Then f is a measurable function.

A. (P. 63)

11. Let f be measurable function finite a.e. on E = [a, b]. Then given $\in > 0, \exists$ a function $\emptyset$, continuous on [a, b] such that:

$$m\,(\{\, x \in E{:}\, f(x) \neq \ \emptyset(x)\}) < \in$$

A. (P. 73)

12. (Weierstrass approximation theorem) If f is a real valued continuous function on [0, 1], then $B_n\,(x) \to f(x)$ uniformly w.r.t. $x\ as\ n\ \to \infty$, where :

$$B_n(x) = \sum_{k=0}^{n} f\,(\frac{k}{n})\, C_k^n\, x^k (1-x)^{n-k}$$

is the Bernstein polynomial of degree n for the function f on [0, 1]

A. (P. 78)

13. The necessary and sufficient condition for a bounded function f defined on the interval [a, b], to be L-integrable over [a, b] is that given $\in > 0$, there exists a measurable partition P of [a, b] such that :

$$U\,(f,P) - \ L\,(f,P) < \in$$

A. (P. 85)

14. Let f and g be two bounded measurable functions defined on a measurable set E, then $f \pm g$ are L-integrable over E and :

$$\int_E \ (f+g)(x)dx = \int_E \ f(x)dx \ \pm \int_E \ g(x)dx$$

A. (P. 93)

15. Let $< f_n >$ be a sequence of bounded measurable functions defined on a measurable set E of finite measure. If there exists a positive number M s.t. $|f_n(x)| \leq M$ for all $n \in N$ and for all $x \in E\ and\ if < f_n >$ converges in measure to a bounded measurable function f on E, then :

$$\lim_{n\to\infty} \int_E \ f_n(x)dx = \int_E \ f(x)dx$$

A. (P. 101)

16. Let $< f_n >$ be a Cauchy sequence of functions in Lebesgue sense over a measurable set E of finite measurable and let $f(x) = \ \lim_{n\to\infty} f_n(x)$ for all $x \ \in E$. Then prove that f is Lebesgue integrable over E and:

$$f(x) = \ \lim_{n\to\infty} \int_E |f_n(x) - \ f(x)|dx = 0$$

A. (P. 107)

17. If f is a non-negative measurable function on a measurable set E and λ is a real number then :

$$\int_E \ \lambda\, f(x)dx = \ \lambda \int_E \ f(x)dx$$

Further if f is summable on E, the λf is also summable.

A. (P. 112)

18. Let $\{f_n\}$ be a sequence of measurable functions converging in measurable to f. If there exists a non-negative summable function $\emptyset$ such that $|f_n(x)| \leq \emptyset(x)$ a.e. on E for each $n \in N$ then :

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$$

A. (P. 121)

19. Let $< f_n >$ be a sequence of summable functions on a set E with finite measure. If $< f_n >$ converges in measure to f and if the family of integrals of f_n is absolutely equi-continuous then f is summable on E and

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$$

A. (P. 123)

20. Prove that L_2 is a complete space.

A. (P. 129)

21. Let $\{f_n\}$ be sequence of functions in L_2 converges in norm to f. Then for any $g \in L_2$ show that :

$$\lim_{n\to\infty} \int_a^b f_n(x)g(x)dx = \int_b^a f(x)g(x)dx$$

A. (P. 132)

22. A series $\sum_{i=1}^{\infty} f_i$ of pairwise orthogonal elements in L_2 is convergent iff the series of real numbers $\sum_{i=1}^{\infty} \|f\|^2$ is convergent.

A. (P. 136)

23. State and prove Bessel's inequality in L_2.

A. (P. 138)

24. Let $\{\emptyset_i\}$ be an orthonormal system in L_2 and $\{a_i\}$ is convergent then $\exists$ a function $f \in L_2$ such that $\|f\|^2 = \sum a_i^2$ where:

$$a_i = < f, \emptyset_i >, \qquad \forall\, i \in N$$

A. (P. 143)

25. Let $\{\emptyset_i\}$ be a complete orthonormal system of functions. of $\{\Psi_i\}$ is a system of functions in L_2 such that :

$$\sum_{i=1}^{\infty} \int_b^a [\emptyset_i(x) - \Psi_i(x)]^2\, dx \leq 1$$

Then prove that the system $\{\Psi_i\}$ is also complete.

A. (P. 146)

26. State and prove Parseval's identity in L_2.

A. (P. 146)

27. Every pth power of summable function on set E is summable on E i.e. $L^p[E] \subset [E]$ but the converse is not true.

A. (P. 148)

28. State and prove holder inequality in L^P space.

A. (P. 150)

29. State and prove Minkowski's inequality in L^P space.

A. (P. 152)

30. Show that space L^P is complete for $p \geq 1$.

A. (P. 154)

31. Prove that the sequence of functions in L^P space has at most one limit.

A. (P. 158)

32. Let X be any set and C is a family of subsets of A which satifhy the properties:
    (i) $\emptyset \in C, X \in C$
    (ii) C is closed under arbitrary intersections.
    (iii) C is closed under finite unions.

    Then ∃ a family of closed subsets of (X, τ)

A. (P. 165)

33. State and prove characterization of a topological space in terms of neighburhoods.

A. (P. 169)

34. Show that a subset A of a tipological space (X, τ) is closed iff $\overline{A} = A$.

A. (P. 181)

35. Let (γ, τ_y) be the subspace of a topological space (X, t) then show tat every Ty-open set in also τ-open iff Y is τ-open.

A. (P. 182)

36. In any topological space, prove that $b\ (A) = \emptyset$ iff A is both open as well as closed.

A. (P. 183)

37. Show that second countable space is always first countable but converse is not true.

A. (P. 187)

38. A one-one onto map $f : (X, \tau) \rightarrow (\gamma, \tau)$ is a homeomorphism iff $f(A) = f(\bar{A})$, for any $A \subset X$.

A. (P. 194)

39. Show that the identity function $I\,; (X, \tau) \rightarrow (X, \tau^*)$ is continuous iff τ is finer than τ^*.

A. (P. 198)

40. Consider the discrete topology D on $X = \{1, 2, 3, 4, 5\}$, find a subspace S of D which does not contain any singleton set.

A. (P. 198)

41. A topological space (X, τ) is a T_0 −space if for any distinct arbitrary points $x, y \in X$, the closure of singleton set {x} and {y} are distinct.

A. (P. 200)

42. Let (X, τ) be any topological space and Let (γ, U) be a Hausdroff space. Let f and g be continuous mappings of X into Y.

A. (P. 206)

43. A topological space is regular iff the collection of all τ-closed nbds form a local base at $x \in X$.

A. (P. 210)

44. A topological space (X, τ) *is a* normal space iff for any closed set F and an open set G containing F, there exist on open set V s.t. $F \subset V \subset \bar{V} \subset G$.

A. (P. 215)

45. Prove that every second countable regular space is normal space.

A. (P. 216)

46. Let Y be a subspace of a topological space (X, τ) *and A is a subset of Y. Then* A is compact relative to X iff A is compact relative to Y.

A. (P. 219)

47. A subset of (R, U) s compact iff it is bounded and closed.
A. (P. 222)
48. In a HAusdrorff topological space disjoint compact sets can be separated by disjoint open sets.
A. (P. 227)
49. If f be a mapping of locally compact space X onto a hausdorff space Y such that f is continuous as well as open, then Y is locally compact.
A. (P. 231)
50. Show that the intersection of the members of an arbitrary family of closed and compact subsets is also closed and compact.
A. (P. 231)
51. Show that T_∞ is a topology on X_∞.
A. (P. 233)
52. Let (X_∞, T_∞) be the one point compactification of a topological space (X, τ) then X_∞ is a Hausdorff space iff X is hausdroff and locally compact.
(P. 235)
53. Let R be the set of all real numbers. Show that the set $S = \{(x, y\}: x^2 + y^2 = 1\}$ in R^2 is the one-point compactification of R and that $\infty = (0, 1)$ is the point at infinity.
A. (P. 236)
54. Show that one point compactification of set of rational numbers Q is not Hausdorff.
A. (P. 238)
55. Let (X, τ) be a topological space and Let (X_∞, T_∞) be its one-point compactification, then X is a dense subset of X_∞ iff X is not compact.
A. (P. 238)
56. A topological space X is disconnected iff ∃ a proper subset of X which is both open and closed in X.
A. (P. 243)
57. Show that closure of a connected set in connected.
A. (P. 244)
58. Union of arbitrary family of connected subset of a topological space is connected if family has non-empty intersection.
A. (P. 249)
59. A subset of R is connected iff it is an interval.
A. (P. 249)
60. A topological space (X, τ) is locally connected iff the components of every open subspace of X are open in X.
A. (P.)
61. Let (X, τ) and (Y, v) be two topological spaces. Then the collection B of Cartesian products of τ-open sets and V-open sets is a base for some topology for Cartesian product $X\ x\ Y$.
A. (P. 255)
62. The product space $(X \times Y, Q)$ is compact iff each of the spaces (X, τ) and (Y, v) is compact.
A. (P. 260)
63. The product space $X = \{X_A : \lambda \in A\}$ is Hausdorff iff each space X_λ is Hausdorff.
A. (P. 264)
64. A topological space is compact iff each finitely short subfamily of sub-basic open sets is short.

A. (P. 267)

65. Let (X, τ) and (Y, v) be two topological spaces. Let f be a continuous mapping of X onto Y such tat f is eiter open or closed, then v must be quotient topology for Y. (i.e. V = τ_f).

A. (P. 270)

66. Let X be a topological space and x/R be a quotient space. If X is compact and connected then X/R is also compact and connected.

A. (P. 272)

67. Let Y be a subset of topological space (X, τ) then a point $X_0 \in X$ is an accumulation point of Y iff ∃ a net in Y – ξ $\{X_0\}$ converging to the point X_0.

A. (P. 279)

68. A topological space is Hausdorff off every net in the space converge to at most one point.

A. (P. 280)

69. Show that every subnet of an ultranet is an ultranet.

A. (P. 291)

70. Let X be a topological space and let $x \in X$. Then show that local base B(x) at x is a filter on X.

A. (P. 291)

71. Let C be any non-void family of subsets of a set X. Then ∃a filter base on X containing C iff C has FIP.

A. (P. 286)

72. A topological space X is hausdorff space iff every convergent filter on X has a unique unit.

A. (P. 289)